Calendars

Chris Williams
Student 1607421
HET 607

21st November 2003

Introduction

There is a plethora of calendars that are currently in use and an even larger number that are no longer in use. For the most part calendars have derived from astronomical events and have evolved over time, through trial and error, to align themselves with astronomical observation. Perhaps the most common alignment is between a calendar and the seasons falling within each year. This paper addresses the difficulties inherent in trying to align a calendar of days with the solar year and the ways that this problem has been addressed through history.

Years, Days, and Months

A fundamental problem that faces the designer of any calendrical system is the awkward arrangement between the various cycles we experience on Earth. Perhaps the most obvious cycle is the day as defined by a cycle of light and dark. The modern definition of a day, the solar day, is the time between successive passes of the Sun directly overhead (zenith crossings). By convention, we take a solar day to have 24 hours each of 60 minutes of 60 seconds, but this is an average and any given solar day may vary a little from this.

After the Sun, the next most dominant feature in the sky is the Moon. Little observation is required to note that its appearance changes with time in a cyclical fashion. The phases of the Moon, full through to new and back to full, provide a convenient measure of the passing days. The cycle occurs every 29.53 days although less precise, early measurements were typically 29 or 30 days. This period is called the lunar or synodic month.

To any early civilisation away from the equatorial regions the next most obvious cycle will be that of the seasons and the year. This cycle is vitally important to agrarian societies because it dictates the most profitable time to sow and harvest crops, when prey or domestic animals might reproduce, and when other natural events, such as the flooding of the Nile river, might occur. The position of the Sun in the sky varies markedly during the course of an annual cycle, with the Sun rising highest above the horizon during summer and lowest during winter. Careful observers could define the length of the cycle by counting days between the time of maximum northerly (or southerly) excursion of the Sun, known as solstices. Alternatively, two days each year the Sun rises due east (the equinoxes) and these can also be used as a datum. Without accurate timekeeping and measurement the result is generally 365 days in the cycle. Modern measurement of the time between vernal equinoxes averages a period of 365.2424 days. The astronomical definition of a year, the mean tropical year, is the average time period between successive vernal equinoxes, spring equinoxes, and summer or winter solstices, and is calculated as 365 days, 5 hours, 48 minutes, and 46 seconds (365.2422 days). The difference, amounting to a day every decade or so, arises because the Earth's orbital motion is not uniform and the orbital plane changes orientation slowly with time. The mean tropical year is often used in calendrical calculations but the vernal equinox year is more correct [1] because many calendars are anchored to the vernal equinox. Throughout this discussion I will use the vernal equinox year of 365.2424 days.

It is clear that there is no neat relationship between days, months and the length of a year. In hindsight there is no reason to expect that there would be, as the axial rotation and orbital motion of the Earth about the Sun are largely independent motions. The motions of the Moon about Earth are similarly independent of its motion about the Sun.

Aligning the Cycles

Three general categories of astronomically-based calendar have been used by cultures throughout history. The categories are:

- Lunar calendars. Lunar calendars are based on the lunar monthly cycle of phases. A calendar year is usually 12 lunar months of thirty days. The difference between the calendar and vernal equinox years meant that seasonal events did not recur on the same date each year.

- Solar calendars. Solar calendars are based solely on the motions of the Sun and the changing of seasons.

- Lunisolar calendars. Lunisolar calendars are a hybrid of the other two types, with months based on the lunar cycle and extra, intercalary, months added periodically to realign the calendar with the seasons.

The ancient Babylonian civilisation is credited with the first codified system of law. The laws included regulation of trade and contract, so the Babylonians also needed a means of reckoning the date for the purposes of record keeping. It was also be convenient if the calendar was useful for agricultural events, sowing, harvesting etc. The first aim is served by a simple set of rules dictating a repetitive cycle. The second aim required careful design of the rules so that the average length of a calendar year matched the vernal equinox year (seasons) to a reasonable accuracy.

The ancient Babylonian solution to calendar design was a lunisolar calender of twelve months, alternating twenty-nine and thirty days for a total of 354 days per year. The alternating months give an average month length of 29.5 days, roughly in line with the lunar cycle at 29.53 days. Alignment of the lunar calendar to the solar cycle was achieved by inserting leap-months, called intercalary months. The base calendar year is 11.2424 days short of a vernal equinox year, a difference that amounts to about a month every three years.

A naïve approach to realigning with the solar cycle would insert an extra 33 or 34 day month every three years, making the error 0.73 or 0.27 days in three years. The extra-long month doesn't sit well with the alternating 29/30 pattern or lunar phases, but it does address the problem of the solar alignment. The Babylonians, however, were more complex in their approach. Early in their history, they calculated that an extra three months were required in an eight year cycle to reasonably align with both the seasons and the Moon. This gives a calendar cycle of $354 \times 8 + 3 \times 30 = 2922$ days or an average year of 365.25 days. The average month was 29.52 days, closer to the true lunar month. Eight vernal equinox years are 2921.94 days, making the calendar discrepancy about 1 day every 16 cycles (or 128 years).

Insertion of intercalary months to realign the calendar with the seasons was a haphazard affair until standardised in approximately 403 BCE. There is evidence that, during the reign of Nabonassar (747-734 BCE), Babylonian astronomers had discovered that 19 solar years were almost exactly 235 lunar months [2] i.e. $19 \times 365.2424 \simeq 235 \times 29.53$ days. Recognition of the cycle lead to the ultimate form of the calendar in which intercalary months were inserted after the month named Addara in years 3, 6, 8, 11, 14, and 19 or Ululu in year 17. This correction gives a calendar cycle of $19 \times 354 + 7 \times 30 = 6936$ days, or an average year and month of 365.05 and 29.51 days respectively. This calendar is, surprisingly, less accurate than its predecessor, being about 3.6 days out over 19 years. While there are 235 months in the 19 calendar years, the months are only an approximation of the true lunar cycle, and the differences mount over the nineteen year period. It is, however, a better calendar from the viewpoint that the rules for intercalating months were fixed. The 19 year cycle later came to be known as the Metonic Cycle after Greek astronomer Meton described it.

A close derivative of the Babylonian calendar is still in use today in the form of the Hebrew calendar. This calendar is used primarily for religious observance in the Jewish faith but is also used by the State of Israel. The Jewish calendar attempts to satisfy the lunar cycle, the solar cycle, and a range of religious constraints and provides a good examples of the difficulties involved. Jewish calendar years consist of twelve lunar months except in years 3, 6, 8, 11, 14, 17 and 19 of a 19 year cycle which contain thirteen (See Table 1). A year may consist of 353, 354, 355, 383, 384, or 385 days. Years with 353 days are called 'defective', with 354 days 'regular, and with 355 days 'complete', and each type has a leap-year equivalent. Defective years are used to ensure certain religious events don't fall on particular days of the week and are always followed by a complete year to regain the lost day. Despite its complexity this calendar is no more accurate than the Babylonian calendar discussed above.

Month	Days	Month	Days
Tishri	30	Nisan	30
Heshvan	29 or 30	Iyar	29
Kislev	30 or 29	Sivan	30
Tevet	29	Tammuz	29
Shevat	30	Av	30
Adar	29 or 30	Elul	29
Adar II	29, leap years only		

Table 1: Hebrew Calendar. Months alternate 30/29 days except for Kislev, which has 29 days in a defective year, and Heshvan, which has 30 days in a complete year. Seven times in each nineteen year cycle the month Adar becomes Adar II and a thirty day intercalary month is inserted as Adar [3].

The ancient Greeks used a similar calendar to their Babylonian contemporaries. However, in keeping with the city-state structure of ancient Greek society, each region had its own variation on the same basic theme. The Athenian calendar is perhaps the best known, and seems to be a rather unsystematic with intercalary months largely added in ad-hoc fashion. The Athenians also maintained a second, variable structure calendar, the conciliar calendar, for political purposes. The conciliar calendar divided the year into periods equal in number to the voting bodies within the society. Unfortunately for historians the conciliar year was neither the same length as the lunisolar calendar year nor fixed in length. After 432BCE, when Meton established his cycle, Greek astronomers tended to use a separate, more regular calendar with extra months in years 3, 5, 8, 11, 13, 16, and 19 of the cycle. This variety of Greek calendrical systems were in use until adoption of the Julian calendar in 42BCE.

Contemporary with the ancient Greek civilisation was that of the Romans. The Roman republican calendar was also of lunisolar construction. The year was 355 days, broken into twelve months of between 28 and 31 days each. The months were no longer tied strongly with the lunar cycle, with month lengths determined by convention rather than observation of the Moon. However, there were defined reference points within each month related to the Moon's phase; the *Kalends*, *Ides*, and *Nones*; from which the day-of-month were counted. With ten days difference to a true solar calendar the republican calendar quickly dropped out of synchronisation with the seasons. When an extra month was required for alignment the month of February would be shortened to 23 or 24 days (from 28) and an extra month of 28 days inserted. The insertion of intercalary months was dictated by a political appointee, the *pontifex maximus* (high priest), often for purely political reasons such as extending a term of office. Consequently, the calendar sometimes slipped seriously out of line with seasons and large, irregular corrections would be required.

On the opposite side of the Mediterranean, the Egyptian civilisation had devised a solar calendar for use in the Nile valley. Egypt was strongly dependent on the annual flooding of the Nile for its agricultural output. Careful observers noted that Sirius, the brightest star in the sky, rose with the Sun around the time of the river flooding. Anchored roughly to the rising of Sirius, the year consisted of twelve, thirty day months followed by five days extra days, 365 days total, to bring the year close to truly solar. Three seasons, Inundation, Emergence, and Low Water of four months each were defined, and reflect the state of the river. This calendar slips by approximately one day every four years, a deficiency corrected during the reign of Ptolemy III (circa 283BCE) through the addition of a leap day every four years. The use of leap-days rather than leap-months to align the calendar with the seasons is one of the major differences between solar and lunar calendars.

Julius Caesar was elected to the post of *pontifex maximus* in 63BCE [4] and later became dictator. By this time the Roman month of January was falling in autumn. Caesar ordered a traditional intercalary month in 47BCE, and again in 46BCE with a further 67 days to correct the alignment [5]. At this juncture the length of the months was increased to give the values recognisable today, and a year of 365 days. Having aligned the calendar, Caesar borrowed the concept of more regular leap-days from the Egyptians. An extra day was inserted between 23 and 24 February every fourth year and intercalated months were no longer used. The choice of insertion point for the extra day was a historical follow-on from the intercalation rule of the republican calendar. The calendar's cycle length is 1461 days, and the average year of 365.25 days gives a slippage of one day every 128 years. This calendar became known as the Julian Calendar.

With the decline of the Roman empire, the Julian calendar was carried forward by the rising Christian church. Amongst the most holy days in the Christian faith is that of the resurrection of the Saviour at Easter. The Julian calendar started each year in January, but the church used Easter as its reference and adopted March as the start of a year. Nevertheless, the date of Easter was notoriously difficult to reach agreement on due to conflicting references in the Bible (e.g the day of the last supper in Matthew 26:17 vs John 18:28), reluctance to base anything on the Jewish definition of Passover (Nisan 15), and the practical problem of working with the Jewish lunar calendar which was yet to be fixed by rule rather than observation. Eventually a reasonable degree of agreement on rules for calculating Easter was reached. The calculation involves an ecclesiastical lunar calendar that approximates the true lunar phases, the date of the vernal equinox, and a conversion into the Julian calendar. The precise date of the vernal equinox was not well known and the Romans held to a traditional date of 25 March while eastern churches used a date derived by observation in Alexandria, 21 March.

As with all previous calendrical systems the difference between calendar and solar years accumulated over time. Over the course of 1600 years since the Julian calendar was introduced the difference amounted to twelve days. Further, the date of Easter on the Julian calendar, which was calculated using a notional lunar calendar, slipped slowly out of synchronisation with the actual phases of the Moon. By 1582 Easter was ten days later than the full Moon, which prompted Pope Gregory XIII to appoint a committee to address the problem. A new calendar was proposed and adopted in 1582: Ten days were dropped to realign Easter to the lunar cycle (October 15 followed October 4). The year was not adjusted by twelve days to correct solar alignment because the religious aspects of Easter were considered to outweigh astronomical aspects. To suppress further misalignments, the leap year rule was adjusted to omit three of four leap-days falling on century years i.e. 1700, 1800, 1900 were not leap years and 1600 was. The rules for the calculation of Easter were also slightly adjusted. This arrangement gives 97 leap years in every 400 for $365\frac{97}{400} = 365.2425$ days in an average year. The new calendar came to bear the name Gregorian.

Despite the rapid introduction of the Gregorian calendar in Rome, religious and political tensions delayed implementation of Gregorian calendar elsewhere in Europe, most of which adhered to the Julian calendar. Dates of implementation vary quite substantially, with Catholic states generally adopting within five years and western protestant states in the eighteenth century [6]. The Gregorian calendar was not adopted in protestant England until 1752, by which time the British had to drop eleven days (Sept. 14 followed Sept. 2) in order to align their calendar [7]. States holding to eastern forms of Christian belief did not, in general, adopt the Gregorian calendar until the twentieth century (Turkey, Greece, Russia). In general the conversion from Julian to Gregorian calendrical system was performed with a one-time adjustment. However, in Sweden an attempt was made to spread the necessary adjustments over an extended period by omitting one day a year or by skipping leap days [8]. This attempt failed during implementation, and Sweden adopted the calendar in 1753 with an eleven day adjustment. With the widespread adoption of the Gregorian calendar in Europe and western Asia its place as a centrepiece of the global financial system was ensured. The Gregorian calendar is now universally used for business purposes.

The near universality of the Gregorian calendar in international affairs has not (yet) meant the demise of other systems. The Hebrew calendar, discussed earlier, is still actively used used for Jewish religious observance. The Chinese have operated a lunisolar calendar since the fourteenth century BCE. The festive calendar is still in use today, although as an adjunct to the Gregorian calendar used for civic duties. The Chinese calendar, while developed separately, shares many traits with the modern Hebrew Calendar. Years are ordinarily 12 months with 353, 354, or 355 days, and leap-years of 13 months with 383, 384, or 385 days. The numbered months are bounded by two new moons and leap-years arise if thirteen new moons occur between month eleven (containing winter solstice) and the following month eleven.

During the eleventh century the Persian calendar, still in use in Iran and Afghanistan, was devised. As designed, the calendar is strictly solar and starts with the vernal equinox, with years of 365 or 366 days depending only on the number of days between equinoxes. Rather than observe the equinox a complex set of rules governing leap years has become popular. A cycle of 2820 years is defined consisting of 21 sub-cycles of 128 years and a final sub-cycle of 132 years. Each sub-cycle is divided into a 29 year group and three 33 year groups except the final group of the last sub-cycle which is extended by four years to 37. Years are numbered from zero within their group and leap-years fall when the year number is divisible by four and is not zero. This rule gives 683 leap-years every 2820 years: an average year of $365\frac{683}{2820} = 365.242199$ days. This figure is remarkably close to the mean tropical year but further from the vernal equinox year than the Gregorian calendar.

There is an alternate approach to aligning a calendar with the annual solar cycle that has not yet be discussed. Several cultures have elected to do nothing to align their calendar to the Sun's cycle. The modern Muslim calendar is a good example: the calendar is strictly lunar with alternating 29 and 30 day months and a 354 day year. The Muslim holy months may fall at any time of year as the lunar calendar shifts with respect to the solar cycle. This calendar, handed down by the Prophet Muhammad, is used for religious observance but most Muslim states also operate a solar calendar for civic organisation purposes.

Another 'calendar' that dispenses with the concept of years and months entirely is, perhaps paradoxically, used by astronomers. The Julian Day Number (JD) is a simple count of days since Noon 1 January 4713 BCE (on the Julian calendar). The JD device was introduced to astronomy by John Herschel (1792-1871) in his 1843 book *Outlines of Astronomy* as a way to unify various archaic calendrical systems in use in astronomical circles. The system is still in use today.

Improving the Calendar

Despite its widespread use, the Gregorian calendar does still not perfectly represent the solar year. The average length of a Gregorian year over its four hundred year cycle, including 97 leap-days, is $365\frac{97}{400} = 365.2425$ days. The difference between this value and the mean tropical year year amounts to approximately one day every 3300 years, and is an even better match to the vernal equinox year. This is an exceptionally good match by past standards, and is dictated by simple rules, but is still seen as open to improvement.

John Herschel suggested the omission of the leap day if the year was also divisible by 4000 [9, 10]. This correction would give 969 leap-days every 4000 years and an average year of 365.24225 days, which an improvement over the existing Gregorian rule if the intent is to match the mean tropical year. It would take approximately 20000 years for a day of misalignment to accumulate. During the French revolution a new calendar was introduced in an attempt to separate the State from religious underpinning of the Gregorian calendar. This calendar made use of a 4000 year rule but the short life of the calendar (1793 to 1806) [9] made it rather a moot point.

The Orthodox Greek Church, when it considered adoption of the Gregorian calendar in the 1920s decided to try and improve the leap-year rules.

They replaced the "divisible by 400" rule with "Every year which when divided by 900 leaves a remainder of 200 or 600 is a leap year." [11] This gives 218 leap-days every 900 years for an average year of 365.24222 days. While this method provides a better match to the mean tropical year than the Gregorian calendar the Greek State adheres to the Gregorian leap-year rule.

One approach to determining better leap-day rules makes use of a mathematical method called continued fractions which gives integer fraction approximations to arbitrary real numbers. For example, 365.2422 days can be approximated by:

$$365.2422 \simeq 365 + \frac{1}{4 + \frac{1}{7 + \frac{1}{1 + \frac{1}{3} \cdots}}} = 365 \frac{31}{128}$$

which suggests that rule giving 31 leap-days every 128 years would be a good match. Such a scheme was proposed by M.B. Cotsworth (1859-1943) with a rule dictating a leap day in years divisible by 4 but not by 128. Analysis of this rule shows an astounding 80000 years to accumulate one day of error.

It is worth noting, however, that over periods of thousands of years a slow change in the length of a the various year measurements becomes significant. The Earth's orbit of the Sun is affected by all the bodies of the solar system in ways that are not resolvable analytically. Consequently, the precise length of a year may not be accurately predicted. Doggett [3] provides the following expression for approximating the length of the tropical year at an arbitrary date:

$$Y = 365.2421896698 - 0.00000615359T - 7.29 \times 10^{-10}T^2 + 2.64 \times 10^{-10}T^3 \text{days}$$

where where T = (JD - 2451545.0)/36525 and JD is the Julian day number. In the year 5000 (JD=3547272.5), the tropical year will be approximately 365.2424 days. Refining a calendar using a fixed notion of the year to which alignment is being made is probably fruitless beyond a few thousand years. A similar variation in the synodic month is given by the formula:

$$M = 29.5305888531 + 0.00000021621T - 3.64 \times 10^{-10}T^2 \text{days}$$

with similar implications for lunar calendars. Calendars could be adjusted to follow these equations but this is not productive. These equations are approximations and are themselves subject to periodic adjustment. Further, of all the calendrical systems outlined in the paper, none has remained unchanged for periods extending more than thousand years or so. This life expectancy of the typical calendar does not, therefore, warrant substantial effort to correct it for longer periods.

Conclusion

A year is not an even number of days, and there's some confusion over the precise definition of a year applicable to calendars in any case. Calendars that work on whole days must be adjusted periodically in order that given seasonal events, such as harvest, religious rites, or floods, fall on or about the same calendar date from year to year. For cultures using the lunar month as the basis for subdivision of the year adjustments are made using extra months, intercalary months, inserted either as-required or in a cyclical pattern. The nineteen year Metonic cycle is perhaps the most used scheme for intercalation of lunar months. For cultures attempting to maintain a solar calendar, adjustment of the calendar is performed using extra days, leap-days, inserted approximately every four years. The Gregorian scheme for leap-day insertion is the most widespread but by no means the only rule set for leap-days. Some calendrical systems chose to completely ignore the solar cycle.

The Gregorian calendar is now firmly entrenched on a global scale. While further improvement is possible, and proposed from time to time, the benefits are small compared to the cost of change. Natural variations, that are not entirely predictable, in the length of the vernal equinox and mean tropical years limit the projection of accuracy to a few thousand years. History tells us that few calendars survive this long without change for pragmatic, political, or other reasons.

References

[1] S. Cassidy. Error in statement of tropical year. Internet `http://home.earthlink.net/~scassidy/`, October 1996.

[2] K. T. Hagen. A History of the Western Calendar. Internet `http://www.polysyllabic.com/CalHist.html`, February 1998.

[3] L. E. Doggett. *Explanatory Supplement to the Astronomical Almanac*, chapter 12. Calendars. University Science Books, Mill Valley, CA, 1992.

[4] Caesar, Julius. In *The Columbia Encyclopedia*. Columbia University Press, 2003.

[5] K. T. Hagen. The Julian Calendar. Internet `http://www.polysyllabic.com/Julian.html`, June 2002.

[6] Richard A. Jarrell. Calendar in the Middle East and Europe. In *Encyclopedia of Astronomy and Astrophysics*. Nature Publishing Group, 2001.

[7] Gregorian Calendar. In *Wikipedia, the free encyclopedia*. Internet http://en.wikipedia.org/wiki/Gregorian_Calendar.

[8] Swedish Calendar. In *Wikipedia, the free encyclopedia*. Internet http://en.wikipedia.org/wiki/Swedish_Calendar.

[9] C. Tøndering. Frequently asked questions about calendars. Internet http://www.tondering.dk/claus/calendar.html.

[10] J. R. Stockton. Leap years. Internet http://www.merlyn.demon.co.uk/leapyear.htm.

[11] G. Lardas. On the question of the "Revised Julian Calendar". Internet http://www.stjohndc.org/what/9609ca1.htm, September 1996.